

**Original Contribution****THE NUMERICAL SOLUTION OF THE EQUATION**  $\frac{\partial U(x,t)}{\partial t} = c \frac{\partial^2 U(x,t)}{\partial x^2}$ **Edibe Elcin<sup>1</sup>, Cengiz Dane<sup>2</sup>**<sup>1\*</sup> Technical Science Vocational Higher school, Trakya University, 22030, Edirne, Turkey<sup>2</sup> Department of Mathematics, Trakya University, Edirne, Turkey**ABSTRACT**

In this study, the solution of non-linear heat transfer equation  $U_t(x,t) = cU_{xx}^{1+a}(x,t)$  with Implicit (Crank-Nicolson's) methods have been studied.

**Key Words:** non-linear heat transfer equation, Implicit (Crank-Nicolson's) methods

**INTRODUCTION**

$$U_t(x,t) = cU_{xx}^{1+a}(x,t) \quad (1)$$

The solution of the non-linear equation in (1) is connected with searching heat transfer as theoretical in material which is irradiated by laser<sub>1</sub>.

This equation is linear parabolic equation, for  $a=0$  and there is analytic solution [3]. If  $a \neq 0$ , this equation is non-linear parabolic equation and there isn't analytic solution [9, 10, 12]. Therefore, approximate solution of this equation is found by numerical methods [1].

In the solution of differential equation which is made by finite difference methods, we consider the following non-dimension form

$$U_t(x,t) = cU_{xx}(x,t) \quad (2)$$

In the solution of boundary-value problem, the explicit and implicit methods is applied [5,14].

Let two dimension boundary-value problems in D be as follows

$$\begin{aligned} Lu &= f \\ u &= u(x,t) \end{aligned} \quad (3)$$

Where L is operator linear or non-linear second order partial differential derivative.

Let  $P_{i,j}(x,t)$  be a point which is defined by  $h = \Delta x, k = \Delta t$  in D. In neighbourhood of  $(x,t)$ , if  $u(x + \Delta x, t)$  and  $u(x - \Delta x, t)$  is opened series and it is arranged again, we have forward difference equation

$$u_x|_{i,j} = 1/h[u_{i+1,j} + u_{i,j}] + O(h) \quad (4)$$

Back forward difference equation

$$u_x|_{i,j} = 1/h[u_{i+1,j} - u_{i,j}] + O(h) \quad (5)$$

center difference equation

$$u_x|_{i,j} = 1/2h[u_{i+1,j} - u_{i-1,j}] + O(h) \quad (6)$$

and

$$u_{xx}|_{i,j} = 1/h^2[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + O(h^2) \quad (7)$$

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where  $O(h)$  is truncation error of approach.

By using second derivative, let us find that the solution of heat transfer equation with implicit methods for  $a = 0, a = -0.2, a = 0.37$  [3,8]

$$\begin{aligned}
 U_t(x,t) &= cU_{xx}^{1+a}(x,t) \\
 u(x,0) &= \text{constan } t, \quad 0 \leq x \leq 1, \quad t = 0 \\
 u(0,t) &= kt, \quad t > 0
 \end{aligned}
 \tag{8}$$

**THE IMPLICIT SOLUTION FOR  $a = 0$**

In (8), if we take  $a = 0$ , we have

$$U_t(x,t) = cU_{xx}(x,t) \tag{9}$$

Let us solve that the equation linear parabolic partial differential equation in(9) in the following initial –boundary condition:

$$\begin{aligned}
 u(x,0) &= 25, \quad 0 \leq x \leq 1, \quad t = 0 \\
 u(0,t) &= k_1 t, \quad t > 0
 \end{aligned}
 \tag{10}$$

In here,  $c = 1,645 \times 10^{-13}$  and  $k_1 = 13000$ . In the point  $(ih, jk)$  of the  $(x,t)$  plane, if we

$$A = \begin{bmatrix} 2.02 & -0.01 & 0 & 0 & 0 & 0 & 0 \\ -0.01 & 2.02 & -0.01 & \dots & 0 & & \\ 0 & -0.01 & 2.02 & \dots & 0 & & \\ 0 & \dots & \dots & \dots & 0 & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \\ 0 & \dots & \dots & -0.01 & 2.02 & & \end{bmatrix}, \quad U_{j+1} = \begin{bmatrix} u_{1,1} \\ \dots \\ \dots \\ u_{9,1} \end{bmatrix}, \quad b_j = \begin{bmatrix} 50,07922735 \\ 50 \\ 50 \\ \dots \\ \dots \\ 50,25 \end{bmatrix}$$

From (13), we have

$$\bar{U}_{j+1} = A^{-1} \bar{b}_j \tag{14}$$

Because A matrix is regular. Hence for  $i = 1, \dots, 9$  and  $j = 1, 2, 3, 4$  it is calculated the values  $u_{i,1}, u_{i,2}, \dots, u_{i,5}$  [3,8].

**THE IMPLICIT SOLUTION FOR**

$a = -0.2$

In (8) equation, if it is taken  $a = -0.2$ , the equation is written by

apply to (10) implicit finite difference methods, we have

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = (ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}) \tag{11}$$

$$\begin{aligned}
 r &= ck/h^2, \\
 x_i &= 0.1, \quad i = 1, \dots, 9, \quad h = 0.1, \quad k = 607902735.56
 \end{aligned}$$

In point which is defined with  $r = 1/100$ , for  $j = 0$ , from (11) equation, we have

$$\begin{aligned}
 -1/100 u_{i-1,j} + (2+2/100) u_{i,j} - 1/100 u_{i+1,j} = \\
 (1/100 u_{i-1,0} + (2-2/100) u_{i,0} + 1/100 u_{i+1,0})
 \end{aligned}
 \tag{12}$$

If this equation is written for  $i = 1, \dots, 9$ , by being used initial and boundary value, we have

$$A \bar{U}_{j+1} = \bar{b}_j \tag{13}$$

Where

$$U_t(x,t) = cU_{xx}^{0.8}(x,t) \tag{15}$$

This equation is given initial –bound values of non-linear parabolic partial differential equation

for  $c = 1.645 \times 10^{-13}, k_1 = 1300$  by (10).

In the point  $(ih, jk)$  of the  $(x,t)$  plane, if we apply to (15) equation implicit finite difference methods, we have

$$1/k(u_{i,j+1}-u_{i,j})=c/2h^2[(u_{i-1,j+1}^{0.8}-2u_{i,j+1}^{0.8}+u_{i+1,j+1}^{0.8})+(u_{i-1,j}^{0.8}-2u_{i,j}^{0.8}+u_{i+1,j}^{0.8})] \tag{16}$$

In this equation, in point which is defined

$$x_i = 0.1, \quad i = 1, \dots, 9, \quad h = 0.1, \quad k = t = 146842878.12$$

by applying  $u_i = v_i + \varepsilon_i$ , the equation (16) can be linearized. In (16) equation it is taken  $u_{i,j+1}$  to  $u_i$ , we have

$$u_{i-1}^{0.8}-2(u_i^{0.8}+pu)+u_{i+1}^{0.8}+[u_{i-1,j}^{0.8}-2u_{i,j}^{0.8}-pu_{i,j}]+u_{i+1,j}^{0.8}=0 \equiv f_i(u_{i-1},u_i,u_{i+1}) \tag{17}$$

where  $r = ck/h^2$ .

On the other hand, from  $f_i(u_1, u_2, \dots, u_N) = 0, \quad i = 1, \dots, N$ .

We have

$$f_i(v_{i-1}, v_i, v_{i+1}) + \left[ \frac{\partial f_i}{\partial u_{i-1}} \varepsilon_{i-1} + \frac{\partial f_i}{\partial u_i} \varepsilon_i + \frac{\partial f_i}{\partial u_{i+1}} \varepsilon_{i+1} \right]_{u_i=v_i} = 0 \tag{18}$$

By using the equation (18) for  $v_i, u_{i,j+1}$  map and  $j = 0$ , the equation (17) is written by

$$0.8u_{i-1,0}^{-0.2}\varepsilon_{i-1} - 2(0.8u_{i,0}^{-0.2} + p)\varepsilon_i + 2u_{i+1,0}^{-0.2}\varepsilon_{i+1} + [2u_{i-1,0}^{0.8} - 4u_{i,0}^{0.8} + 2u_{i+1,0}^{0.8}] = 0 \tag{19}$$

For  $i = 1, \dots, 9, \quad p = 100$ , in the point  $t = 0$  by considering (10), we solve  $\varepsilon_i$ , then we have

$$\varepsilon_1 = \dots = \varepsilon_9 = 0 \tag{20}$$

For  $i = 1, \dots, 9$ , from  $u_i = v_i + \varepsilon_i$ , we have  $u_{i,1}, u_{i,2}, \dots, u_{i,5}$  [3,8].

**THE IMPLICIT SOLUTION FOR**

$$a = 0.37$$

In the (8) equation, if it is taken  $a = 0.37$ , the equation is written by

$$U_t(x,t) = cU_{xx}^{1.37}(x,t) \tag{21}$$

and  $c = 1.0392 \times 10^{-15}, k_1 = 1300$ . (21) non-linear parabolic partial differential equation initial-boundary condition is given as in (10). In the point  $(ih, jk)$  of the  $(x, t)$  plane, if we apply to (21) equation implicit finite difference methods, we have

$$1/k(u_{i,j+1}-u_{i,j})=c/2h^2[(u_{i-1,j+1}^{1.37}-2u_{i,j+1}^{1.37}+u_{i+1,j+1}^{1.37})+(u_{i-1,j}^{1.37}-2u_{i,j}^{1.37}+u_{i+1,j}^{1.37})] \tag{22}$$

In this equation, in the point which is defined

$$x_i = 0.1, \quad i = 1, \dots, 9, \quad h = 0.1, \quad k = t = 96227867590$$

by applying  $u_i = v_i + \varepsilon_i$ . The equation (22) can be linearized. In (22) equation, if it is taken  $u_i$  instead of  $u_{i,j+1}, p = ck/h^2$  we have

$$u_{i-1}^{1.37}-2(u_i^{1.37}+pu)+u_{i+1}^{1.37}+[u_{i-1,j}^{1.37}-2u_{i,j}^{1.37}-pu_{i,j}]+u_{i+1,j}^{1.37}=0 \equiv f_i(u_{i-1},u_i,u_{i+1}) \tag{23}$$

On the other hand, by using the equation (18), for (23) map  $v_i, u_{j+1}$  and for  $j = 0$ , the equation (23) is written by

$$1.37u_{i-1,0}^{0.37}\varepsilon_{i-1} - 2(1.37u_{i,0}^{0.37} + p)\varepsilon_i + 2u_{i+1,0}^{0.37}\varepsilon_{i+1} + [2u_{i-1,0}^{0.37} - 4u_{i,0}^{0.37} + 2u_{i+1,0}^{0.37}] = 0 \tag{24}$$

For  $i = 1, \dots, 9, \quad p = 100$ , in the point  $t = 0$  by considering initial-bound value, if we solve  $\varepsilon_i$ , then we have

$$\varepsilon_1 = \dots = \varepsilon_9 = 0 \tag{25}$$

In the similarly, for  $i = 1, \dots, 9$ , from  $u_i = v_i + \varepsilon_i$ , we have  $u_{i,1}, u_{i,2}, \dots, u_{i,5}$  [3,8].

As a result of, the solution which is made by implicit method is more convergence than the solution which is made by explicit method. Things that are obtained in apply coincide with experimental examinations.

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