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Original Contribution

THE NUMERICAL SOLUTION OF THE EQUATION $\frac{\partial U(x,t)}{\partial t} = c\frac{\partial^2 U(x,t)}{\partial x^2}$

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ABSTRACT

In this study, the solution of non-linear heat transfer equation $U_t(x,t) = cU_{xx}^{1+a}(x,t)$ with Implicit (Crank-Nicolson's) methods have been studied.

Key Words: non-linear heat transfer equation, Implicit (Crank-Nicolson's) methods

INTRODUCTION

$$U_t(x,t) = cU_{xx}^{1+a}(x,t) \quad (1)$$

The solution of the non-linear equation in (1) is connected with searching heat transfer as theoretical in material which is irradiated by laser$_1$.

This equation is linear parabolic equation, for $a = 0$ and there is analytic solution [3]. If $a \neq 0$, this equation is non-linear parabolic equation and there isn't analytic solution [9, 10, 12].Therefore, approximate solution of this equation is found by numerical methods [1].

In the solution of differential equation which is made by finite difference methods, we consider the following non-dimension form

$$U_t(x,t) = cU_{xx}(x,t) \quad (2)$$

In the solution of boundry-value problem, the explicit and implicit methods is applied [5,14].

Let two dimension boundry-value problems in D be as follows

$$\begin{aligned} Lu &= f \\ u &= u(x,t) \end{aligned} \quad (3)$$

Where L is operator linear or non-linear second order partial differantial derivative.

Let $P_{i,j}(x,t)$ be a point which is defined by $h = \Delta x, k = \Delta t$ in D. In neighbourhood of (x,t), if $u(x+\Delta x,t)$ *and* $u(x-\Delta x,t)$ is opened series and it is arranged again, we have forward difference equation

$$u_x\big|_{i,j} = 1/h[u_{i+1,j} + u_{i,j}] + O(h) \quad (4)$$

Back forward difference equation

$$u_x\big|_{i,j} = 1/h[u_{i+1,j} - u_{i,j}] + O(h) \quad (5)$$

center difference equation

$$u_x\big|_{i,j} = 1/2h[u_{i+1,j} - u_{i-1,j}] + O(h) \quad (6)$$

and

$$u_{xx}\big|_{i,j} = 1/h^2[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + O(h^2) \quad (7)$$

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where $O(h)$ is trancutation error of approach.

By using second derivative, let us find that the solution of heat transfer equation with implicit methods for $a = 0,\ a = -0.2$, $a = 0.37$ [3,8]

$$
\begin{aligned}
&U_t(x,t) = cU_{xx}^{1+a}(x,t) \\
&u(x,0) = constant\, t, \quad 0 \le x \le 1, \quad t = 0 \\
&u(0,t) = kt, \quad t > 0
\end{aligned}
\tag{8}
$$

THE IMPLICIT SOLUTION FOR $a = 0$

In (8), if we take $a = 0$, we have

$$U_t(x,t) = cU_{xx}(x,t) \qquad (9)$$

Let us solve that the equation linear parabolic partial differential equation in(9) in the following initial –boundary condition:

$$
\begin{aligned}
&u(x,0) = 25, \quad 0 \le x \le 1, \qquad t = 0 \\
&u(0,t) = k_1 t\ , \quad t > 0
\end{aligned}
\qquad (10)
$$

In here, $c = 1{,}645 \times 10^{-13}$ *and* $k_1 = 13000$. In the point (ih, jk) of the (x,t) plane, if we apply to (10) implicit finite difference methods, we have

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = \left(ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}\right) \qquad (11)$$

$r = ck/h^2$,

$x_i = 0.1, \quad i = 1,...,9,\ h = 0.1, \quad k = 607902735.56$

.

In point which is defined with $r = 1/100,\ for\ j = 0$, from (11) equation, we have

$$-1/100\ u_{i-1,j} + (2+2/100)\ u_{i,,1} - 1/100\ u_{i+1,j} = \left(1/100\ u_{i-1,0+}(2-2/100)\ u_{i,0} + 1/100 u_{i+ \dots}\right) \qquad (12)$$

If this equation is written for $i = 1,..,9$, by being used initial and boundary value, we have

$$A\overline{U}_{j+1} = \overline{b}_j \qquad (13)$$

Where

$$
A = \begin{bmatrix}
2.02 & -0.01\ 0\,0\,0\ \ 0\,0\ \ 0\ \ 0 \\
-0.01\ \ 2.02 & -0.01..........0 \\
0 \quad -0.01 & 2.02.........0 \\
0 \ldots\ldots\ldots\ldots\ldots & 0 \\
0\ldots\ldots\ldots\ldots & -0.01\ \ 2.02
\end{bmatrix},\quad
U_{j+1} = \begin{bmatrix} u_{1,1} \\ ... \\ .... \\ .... \\ u_{9,1} \end{bmatrix},
b_j = \begin{bmatrix} 50{,}07922735 \\ 50 \\ 50 \\ ... \\ ... \\ 50{,}25 \end{bmatrix}
$$

From (13), we have

$$\overline{U}_{j+1} = A^{-1}\overline{b}_j \qquad (14)$$

Because A matrix is regular. Hence for $i = 1,..,9$ and $j = 1,2,3,4$ it is calculated the values $u_{i,1}, u_{i,2}, ..., u_{i,5}$ [3,8].

THE IMPLICIT SOLUTION FOR $a = -0.2$

In (8) equation, if it is taken $a = -0.2$, the equation is written by

$$U_t(x,t) = cU_{xx}^{0.8}(x,t) \qquad (15)$$

This equation is given initial –bound values of non-linear parabolic partial differential equation

for $c = 1.645 \times 10^{-13}$, $k_1 = 1300$ by (10).

In the point (ih, jk) of the (x,t) plane, if we apply to (15) equation implicit finite difference methods, we have

$$1/k(u_{i,j+1}-u_{i,j})=c/2h^2[(u^{0.8}_{i-1,j+1}-2u^{0.8}_{i,j+1}+u^{0.8}_{i+1,j+1})+(u^{0.8}_{i-1,j}-2u^{0.8}_{i,j}+u^{0.8}_{i+1,j})] \quad (16)$$

In this equation, in point which is defined

$x_i = 0.1, \quad i = 1,...,9, \; h = 0.1, \; k = t = 146842878.12$

by applying $u_i = v_i + \varepsilon_i$, the equation (16) can be lineared. In (16) equation it is taken $u_{i,j+1}$ *to* u_i, we have

$$u^{0.8}_{i-1}-2(u^{0.8}_{i}+pu_i)+u^{0.8}_{i+1}+[u^{0.8}_{i-1,j}-2(u^{0.8}_{i,j}-pu_{i,j})+u^{0.8}_{i+1,j}]=0 \equiv f_i(u_{i-1},u_i,u_{i+1}) \quad (17)$$

where $r = ck/h^2$.

On the other hand, from $f_i(u_1,u_2,..,u_N)=0, \; i=1,...,N$.

We have

$$f_i(v_{i-1},v_i,v_{i+1})+[\frac{\partial f_i}{\partial u_{i-1}}\varepsilon_{i-1}+\frac{\partial f_i}{\partial u_i}\varepsilon+\frac{\partial f_i}{\partial u_{i+1}}\varepsilon_{i+1}]_{u_i=v_i}=0 \quad (18)$$

By using the equation (18) for $v_i, u_{i,j+1}$ map and $j = 0$, the equation (17) is written by

$$0.8u^{-0.2}_{i-1,0}\varepsilon_{i-1}-2(0.8u^{-0.2}_{i,0}+p)\varepsilon_i+2u^{-0.2}_{i+1,0}\varepsilon_{i+1}+[2u^{0.8}_{i-1,0}-4u^{0.8}_{i,0}+2u^{0.8}_{i+1,0}]=0 \quad (19)$$

For $i = 1,...9, \; p = 100$, in the point $t = 0$ by considering (10), we solve ε_i, then we have

$$\varepsilon_1 = ... = \varepsilon_9 = 0 \quad (20)$$

For $i = 1,...9$, from $u_i = v_i + \varepsilon_i$, we have $u_{i,1}, u_{i,2}, ..., u_{i,5}$ [3,8].

THE IMPLICIT SOLUTION FOR $a = 0.37$

In the (8) equation, if it is taken $a = 0.37$, the equation is written by

$$U_t(x,t) = cU^{1..37}_{xx}(x,t) \quad (21)$$

and $c = 1.0392\times10^{-15}, k_1 = 1300$.(21) non-linear parabolic partial differential equation initial –boundry condition is given as in (10). In the point (ih, jk) of the (x,t) plane, if we apply to (21) equation implicit finite difference methods, we have

$$1/k(u_{i,j+1}-u_{i,j})=c/2h^2[(u^{1.37}_{i-1,j+1}-2u^{1.37}_{i,j+1}+u^{1.37}_{i+1,j+1})+(u^{1.37}_{i-1,j}-2u^{1.37}_{i,j}+u^{1.37}_{i+1,j})] \quad (22)$$

In this equation, in the point which is defined

$x_i = 0.1, \quad i = 1,...,9, \; h = 0.1, \; k = t = 96227867590$

by applying $u_i = v_i + \varepsilon_i$. The equation (22) can be lineared. In (22) equation, if it is taken u_i instead of $u_{i,j+1}$, $p = ck/h^2$ we have

$$u^{1.37}_{i-1}-2(u^{1.37}_{i}+pu_i)+u^{1.37}_{i+1}+[u^{1.37}_{i-1,j}-2(u^{1.37}_{i,j}-pu_{i,j})+u^{1.37}_{i+1,j}]=0 \equiv f_i(u_{i-1},u_i,u_{i+1}) \quad (23)$$

On the other hand, by using the equation (18), for (23) map v_i, u_{j+1} and for $j = 0$, the equation (23) is written by

$$1.37u^{0..37}_{i-1,0}\varepsilon_{i-1}-2(1.37u^{0.37}_{i,0}+p)\varepsilon_i+2u^{0.37}_{i+1,0}\varepsilon_{i+1}+[2u^{0.37}_{i-1,0}-4u^{0.37}_{i,0}+2u^{0.37}_{i+1,0}]=0 \quad (24)$$

For $i = 1,...9, \; p = 100$, in the point $t = 0$ by considering initial –bound value, if we solve ε_i, then we have

$$\varepsilon_1 = ... = \varepsilon_9 = 0 \quad (25)$$

In the similarly, for $i = 1,...9$, from $u_i = v_i + \varepsilon_i$, we have $u_{i,1}, u_{i,2}, ..., u_{i,5}$ [3,8].

As a result of, the solution which is made by implicit method is more convergence than the solution which is made by explicit method. Things that are obtained in apply coincide with experimental examinations.

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