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Original Contribution

THE CONVERGENCE OF THE NUMERICAL SOLUTION OF THE EQUATION $\frac{\partial U(x,t)}{\partial t} = c\frac{\partial^2 U(x,t)}{\partial x^2}$

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ABSTRACT
In this study, the convergence of the solution of non-linear heat transfer equation $U_t(x,t) = cU_{xx}^{1+a}(x,t)$ with explicit and Crank-Nicolson's methods have been studied.

Key Words: *non-linear heat transfer equation, Crank-Nicolson's methods*

INTRODUCTION

For $a = 0$, the equation

$$U_t(x,t) = cU_{xx}^{1+a}(x,t) \qquad (1)$$

is linear parabolic equation and there is analytic solution [3].If $a \neq 0$, this equation is non-linear parabolic equation and there isn't analytic solution [9,10,13] 1.

As finding the non-linear equation's solutions as analytic are not possible everytime, it is obtained approach solutions with numerical methods. But, in the solutions which are made by numerical methods, which is used should be convergence and stable so that the solution is meaningful [1].

In the work, in the linear heat transfer equations the numerical solutions which is made by Explicit and Crank-Nicolson methods, it is shown that the solution is convergence, if $a \geq -1$, and $r \leq \frac{1}{2(1+a)U_{i,j}^{a}}$.

NUMERICAL SOLUTION METHODS OF THE LINEAR PARABOLIC EQUATIONS

In heat transfer equation

$$\frac{\partial U(x,t)}{\partial t} = c\frac{\partial^2 U(x,t)}{\partial x^2} \quad (2)$$

for
$x = ih,\ i = 1,...,N,\ t = jk,\ j = 1,...,N\ and\ r = k/h^2$
, if it is approached to $\frac{\partial U(x,t)}{\partial t}$ with forward difference and if it is approached to $\frac{\partial^2 U(x,t)}{\partial x^2}$ with center difference, the equation (2) is written as follows

$$U_{i,j} = U_{i,j} + r(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) \quad (3)$$

The equation (3) is explicit equation of the equation (2) [15]. Likewise, in the equation (2), if it is approached to $\frac{\partial U(x,t)}{\partial t}$ with average of forward difference and if it is approached to $\frac{\partial^2 U(x,t)}{\partial x^2}$ with average of center difference,
for
$x = ih,\ i = 1,...,N,\ t = jk,\ j = 1,...,N\ and\ r = k/h^2$
, the equation (2) is written as follows

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$$-rU_{i-1,j+1}+(2+2r)U_{i,j+1}-rU_{i+1,j+1}=\left(rU_{i-1,j}+(2-2r)U_{i,j}+rU_{i+1,j}\right) \qquad (4)$$

This equation is Crank-Nicolson Equation of the equation (2) [1, 9, 15].

THE CONVERGENCE OF SOLUTION IN THE SOLUTION MADE WITH EXPLICIT METHODS OF THE EQUATION $\frac{\partial U(x,t)}{\partial t}=c\,\frac{\partial^2 U(x,t)}{\partial x^2}$

If it is applied explicit formula to in the equation (1), it is obtained the non-linear explicit finite difference as follows

$$U_{i,j+1}=rU_{i-1,j}^{1+a}-2rU_{i,j}^{1+a}+rU_{i+1,j}^{1+a}+U_{i,j} \qquad (5)$$

Let

$$e=U-u \qquad (6)$$

Where u is a finite difference convergence and U is the analytic solution of equation [5].
Hence, the equation (5) is expressed as follows,

$$\begin{aligned} e_{i,j+1}&=-rU_{i+1,j}^{1+a}+2rU_{i,j}^{1+a}-rU_{i-1,j}^{1+a}+r(1+a)U_{i+1,j}^{a}e_{i+1,j}\\ &+[1-2r(1+a)U_{i,j}^{a}]e_{i,j}+r(1+a)U_{i-1,j}^{a}e_{i-1,j}+U_{i,j+1}-U_{i,j} \end{aligned} \qquad (7)$$

for $0<\theta_1,\theta_2,\theta_3<1$, $-1<\theta_4<1$, the equation(7) is written as follows,

$$\begin{aligned} e_{i,j+1}&=r(1+a)U_{i+1,j}^{a}e_{i+1,j}+[1-2r(1+a)U_{i,j}^{a}]e_{i,j}+r(1+a)U_{i-1,j}^{a}e_{i-1,j}\\ &+k[\frac{\partial U}{\partial t}(xi,tj+\theta_3 k)-\frac{\partial U^{1+a}}{\partial x^2}(xi+\theta_4 h,tj)] \end{aligned} \qquad (8)$$

In here, $U\geq 0$. If

$$E_j=\underset{j}{Max}\left|e_{i,j}\right| \qquad (9)$$

and

$$Max[\frac{\partial U}{\partial t}(xi,tj+\theta_3 k)-\frac{\partial U^{1+a}}{\partial x^2}(xi+\theta_4 h,tj)]=M \qquad (10)$$

The following inequality should be hold

$$[1-2r(1+a)U_{i,j}^{a}]\geq 0 \qquad (11)$$

So that all of terms of the equation (8) is equal or bigger than zero. Hence we have

$$r\leq\frac{1}{2(1+a)U_{i,j}^{a}} \qquad (12)$$

On the other hand, in the equation $r=ck/h^2$, we have $r>0$ because h is step length of x, k is step length of t, c is positive constant. Therefore, we have

$$0<r\leq\frac{1}{2(1+a)U_{i,j}^{a}} \qquad (13)$$

The condition (13) is the convergence criteria of the explicit finite difference equation, for the equation (1).If $a > -1$ then

$$\frac{1}{2(1+a)U_{i,j}^{a}} \tag{14}$$

is positive . Hence if $a > -1$ then the inequality (13) is hold. On the other hand, the equation (8) is written as follows

$$\begin{aligned} |\ e_{i,j+1}\ | &\le r(1+a)U_{i+1,j}^{a}|\ e_{i+1,j}\ | \\ &+ [1-2r(1+a)U_{i,j}^{a}]|\ e_{i,j}\ | + r(1+a)U_{i-1,j}^{a}|\ e_{i-1,j}\ | + kM \end{aligned} \tag{15}$$

For A is constant and $A > 0$, if

$$(1+a)U_{i+1,j}^{a} \cong\ (1+a)U_{i,j}^{a}] \cong U_{i-1,j+1}^{a} \ge A \tag{16}$$

then, the equation (15) is written as follows

$$E_{j+1} \le E_j + kM \tag{17}$$

[1,15].In here, for the equation (17) we have

$$E_{j+1} \le E_j + kM = E_{j-1} + 2kM = ... = E_0 + jkM = tM \tag{18}$$

Therefore,

$$\lim_{\substack{h \to 0 \\ k \to rh^2/c}} M = (U_t - U_{xx}^{1+a})_{i,j} \tag{19}$$

Because the inequality has zero error. U is the analytic solution of $U_t = U_{xx}^{1+a}$, for $M \to 0$, we have

$$E_j \ge |u_{i,j} - U_{i,j}| \quad \text{and} \quad u \to U \tag{20}$$

[10].So, for $a > -1$, the equation is convergence if

$$r \le \frac{1}{2(1+a)U_{i,j}^{a}} \tag{21}$$

THE CONVERGENCE OF THE SOLUTION WITH CRANK-NICOLSON METHODS OF THE NON-LINEAR HEAT TRANSFER EQUATION

If Crank-Nicolson Formula is applied to the equation (1), we have

$$-rU_{i-1,j+1}^{1+a} + 2rU_{i,j+1}^{1+a} - rU_{i+1,j+1}^{1+a} + U_{i,j+1} = rU_{i-1,j}^{1+a} - 2rU_{i,j}^{1+a} + rU_{i+1,j}^{1+a} + U_{i,j} \tag{22}$$

This is non-linear Crank-Nicolson finite difference equation.By using (6), the equation (22) is written as follows

$$[2(1+a)U_{i,j+1}^{a}+1/r]e_{i,j+1}=-U_{i-1,j+1}^{1+a}+2U_{i,j+1}^{1+a}-U_{i+1,j+1}^{1+a}-U_{i-1j}^{1+a}+$$
$$2U_{i,j}^{1+a}-U_{i+1,j}^{1+a}+(1+a)U_{i-1,j+1}^{a}e_{i-1,j+1}+(1+a)U_{i+1,j}^{a}e_{i+1,j+1}+(1+a)U_{i-1,j}^{a}e_{i-1,j} \quad (23)$$
$$-[2(1+a)U_{i,j}^{a}-1/r]e_{i,j}+(1+a)U_{i+1,j}^{a}e_{i+1,j}+(1/r)(U_{i,j+1}-U_{i,j})$$

For $0<\theta_1,\theta_2,\theta_3,\theta_4,\theta_5<1$, $-1<\theta_6<1$, the equation (23) is written as follows

$$[2(1+a)U_{i,j+1}^{a}+1/r]e_{i,j+1}=(1+a)U_{i-1,j+1}^{a}e_{i-1,j+1}+(1+a)U_{i+1,j+1}^{a}e_{i+1,j+1}$$
$$+(1+a)U_{i-1,j}^{a}e_{i-1,j}-[2(1+a)U_{i,j}-1/r]e_{i,j}+(1+a)U_{i+1,j}e_{i+1,j} \quad (24)$$
$$+2h^2[\frac{\partial U}{\partial t}(x_i,t_j+\theta_5 k)-\frac{\partial^2 U^{1+a}}{\partial x^2}(x_i+\theta_6 h,t_j)]$$

In here, $U\geq 0.$ From (9) and

$$Max[\frac{\partial U}{\partial t}(x_i,t_j+\theta_5 k)-\frac{\partial^2 U^{1+a}}{\partial x^2}(x_i+\theta_6 h,t_j)]=M \quad (25)$$

The following inequality is hold

$$2(1+a)U_{i,j}^{a}-\frac{1}{r}\leq 0 \quad (26)$$

So that all of terms of (24) is equal or bigger than zero.If $a>-1$, then

it is true. Therefore, if

$$r\leq\frac{1}{2(1+a)U_{i,j}^{a}} \quad (27)$$

The equation (1) is convergence for Crank-Nicolson finite difference methods.

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